3.3. VECTORS IN THE PLANE
What You Should Learn

- Represent vectors as directed line segments.
- Write the component forms of vectors.
- Perform basic vector operations and represent them graphically.
- Write vectors as linear combinations of unit vectors.
- Find the direction angles of vectors.
- Use vectors to model and solve real-life problems.
Introduction

Quantities such as force and velocity involve both *magnitude* and *direction* and cannot be completely characterized by a single real number.

To represent such a quantity, you can use a **directed line segment**, as shown in the Figure.
The directed line segment $\overrightarrow{PQ}$ has **initial point** $P$ and **terminal point** $Q$. Its **magnitude** (or length) is denoted by $\|\overrightarrow{PQ}\|$ and can be found using the Distance Formula.

Two directed line segments that have the same magnitude and direction are equivalent.

For example, the directed line segments in the Figure are all equivalent.
The set of all directed line segments that are equivalent to the directed line segment $\overrightarrow{PQ}$ is a vector $\mathbf{v}$ in the plane, written

$$\mathbf{v} = \overrightarrow{PQ}.$$  

Vectors are denoted by lowercase, boldface letters such as $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. 
Example

Let \( \mathbf{u} \) be represented by the directed line segment from \( P(0, 0) \) to \( Q(3, 2) \), and let \( \mathbf{v} \) be represented by the directed line segment from \( R(1, 2) \) to \( S(4, 4) \). Show that \( \mathbf{u} \) and \( \mathbf{v} \) are equivalent.
Solution

(Show: magnitude and direction(slope) are the same)

From the Distance Formula, it follows that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the same magnitude.

\[
\| \overrightarrow{PQ} \| = \sqrt{(3 - 0)^2 + (2 - 0)^2} = \sqrt{13}
\]

\[
\| \overrightarrow{RS} \| = \sqrt{(4 - 1)^2 + (4 - 2)^2} = \sqrt{13}
\]

Moreover, both line segments have the same direction because they are both directed toward the upper right on lines having a slope of

\[
\frac{4 - 2}{4 - 1} = \frac{2 - 0}{3 - 0} = \frac{2}{3}.
\]

Because $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the same magnitude and direction, $\mathbf{u}$ and $\mathbf{v}$ are equivalent.
Component Form of a Vector

The directed line segment whose initial point is the origin is often the most convenient representative of a set of equivalent directed line segments.

This representative of the vector \( \mathbf{v} \) is in **standard position**.

A vector whose initial point is the origin \((0, 0)\) can be uniquely represented by the coordinates of its terminal point \((v_1, v_2)\).

This is the **component form of a vector** \( \mathbf{v} \), written as \( \mathbf{v} = \langle v_1, v_2 \rangle \).
The coordinates $v_1$ and $v_2$ are the *components* of $v$.

If both the initial point and the terminal point lie at the origin, $v$ is the **zero vector** and is denoted by $0 = \langle 0, 0 \rangle$.

**Component Form of a Vector**

The component form of the vector with initial point $P(p_1, p_2)$ and terminal point $Q(q_1, q_2)$ is given by

$$\overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle v_1, v_2 \rangle = v.$$  

The **magnitude** (or length) of $v$ is given by

$$\|v\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{v_1^2 + v_2^2}.$$  

If $\|v\| = 1$, $v$ is a **unit vector**. Moreover, $\|v\| = 0$ if and only if $v$ is the zero vector $0$.

Two vectors $u = \langle u_1, u_2 \rangle$ and $v = \langle v_1, v_2 \rangle$ are **equal** if and only if $u_1 = v_1$ and $u_2 = v_2$. 

Example

The vector \( \mathbf{u} \) which is from \( P(0, 0) \) to \( Q(3, 2) \) is \( \mathbf{u} = \overrightarrow{PQ} = \langle 3 - 0, 2 - 0 \rangle = \langle 3, 2 \rangle \). The vector \( \mathbf{v} \) which is from \( R(1, 2) \) to \( S(4, 4) \) is \( \mathbf{v} = \overrightarrow{RS} = \langle 4 - 1, 4 - 2 \rangle = \langle 3, 2 \rangle \).

This is another way to show the vectors \( \mathbf{u} \) and \( \mathbf{v} \) are equivalent.
Example

Find the component form and magnitude of the vector \( \mathbf{v} \) that has initial point \((4, -7)\) and terminal point \((-1, 5)\).
Solution

Let

\[ P(4, -7) = (p_1, p_2) \]

and

\[ Q(-1, 5) = (q_1, q_2). \]

Then, the components of \( \mathbf{v} = \langle v_1, v_2 \rangle \) are

\[ v_1 = q_1 - p_1 = -1 - 4 = -5 \]

\[ v_2 = q_2 - p_2 = 5 - (-7) = 12. \]
Solution

So, \( \mathbf{v} = \langle -5, 12 \rangle \) and the magnitude of \( \mathbf{v} \) is

\[
\|
\mathbf{v}
\| = \sqrt{(-5)^2 + 12^2}
\]

\[
= \sqrt{169}
\]

\[
= 13.
\]
Vector Operations: scalar multiplication

The two basic vector operations are **scalar multiplication** and **vector addition**. In operations with vectors, numbers are usually referred to as **scalars**.

In this section, scalars will always be real numbers. Geometrically, the product of a vector $\mathbf{v}$ and a scalar $k$ is the vector that is $|k|$ times as long as $\mathbf{v}$. 
Vector Operations: scalar multiplication

If $k$ is positive, $kv$ has the same direction as $v$, and if $k$ is negative, $kv$ has the direction opposite that of $v$, as shown in the Figure.

To add two vectors $u$ and $v$ geometrically, first position them (without changing their lengths or directions) so that the initial point of the second vector $v$ coincides with the terminal point of the first vector $u$. 
Vector Operations: Sum

The sum $\mathbf{u} + \mathbf{v}$ is the vector formed by joining the initial point of the first vector $\mathbf{u}$ with the terminal point of the second vector $\mathbf{v}$ after positioning the initial point of the vector $\mathbf{v}$ at the terminal point of the vector $\mathbf{u}$, as shown in the Figure.
Vector Operations:
scalar multiplication and sum

This technique is called the **parallelogram law** for vector addition because the vector \( u + v \), often called the **resultant** of vector addition, is the diagonal of a parallelogram having adjacent sides \( u \) and \( v \).

**Definitions of Vector Addition and Scalar Multiplication**

Let \( u = \langle u_1, u_2 \rangle \) and \( v = \langle v_1, v_2 \rangle \) be vectors and let \( k \) be a scalar (a real number). Then the **sum** of \( u \) and \( v \) is the vector

\[
    u + v = \langle u_1 + v_1, u_2 + v_2 \rangle
\]

and the **scalar multiple** of \( k \) times \( u \) is the vector

\[
    ku = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle.
\]
Vector Operations: Difference

The **negative** of \( v = \langle v_1, v_2 \rangle \) is

\[
-\mathbf{v} = (-1)\mathbf{v} \\
= \langle -v_1, -v_2 \rangle
\]

and the **difference** of \( \mathbf{u} \) and \( \mathbf{v} \) is

\[
\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) \\
= \langle u_1 - v_1, u_2 - v_2 \rangle.
\]
Vector Operations: Difference

To represent $\mathbf{u} - \mathbf{v}$ geometrically, you can use directed line segments with the same initial point.

The difference $\mathbf{u} - \mathbf{v}$ is the vector from the terminal point of $\mathbf{v}$ to the terminal point of $\mathbf{u}$, which is equal to $\mathbf{u} + (-\mathbf{v})$, as shown in the Figure.
Vector Operations

The component definitions of vector addition and scalar multiplication are illustrated in the next Example.

In this example, notice that each of the vector operations can be interpreted geometrically.
Example

Let \( \mathbf{v} = \langle -2, 5 \rangle \) and \( \mathbf{w} = \langle 3, 4 \rangle \), and find each of the following vectors.

a. \( 2\mathbf{v} \) \quad b. \( \mathbf{w} - \mathbf{v} \) \quad c. \( \mathbf{v} + 2\mathbf{w} \)
Solution

a. Because \( \mathbf{v} = \langle -2, 5 \rangle \), you have

\[
2\mathbf{v} = 2\langle -2, 5 \rangle \\
= \langle 2(-2), 2(5) \rangle \\
= \langle -4, 10 \rangle.
\]

A sketch of \( 2\mathbf{v} \) is shown in the Figure.
Solution

b. The difference of \( \mathbf{w} \) and \( \mathbf{v} \) is

\[
\mathbf{w} - \mathbf{v} = \langle 3, 4 \rangle - \langle -2, 5 \rangle
\]

\[
= \langle 3 - (-2), 4 - 5 \rangle
\]

\[
= \langle 5, -1 \rangle.
\]

A sketch of \( \mathbf{w} - \mathbf{v} \) is shown in the Figure.

Note that the figure shows the vector difference \( \mathbf{w} - \mathbf{v} \) as the sum \( \mathbf{w} + (-\mathbf{v}) \).
Solution

c. The sum of \( v \) and \( 2w \) is

\[
v + 2w = \langle -2, 5 \rangle + 2\langle 3, 4 \rangle = \langle -2, 5 \rangle + \langle 2(3), 2(4) \rangle = \langle -2, 5 \rangle + \langle 6, 8 \rangle = \langle -2 + 6, 5 + 8 \rangle = \langle 4, 13 \rangle.
\]

A sketch of \( v + 2w \) is shown in the Figure.
Vector Operations

Vector addition and scalar multiplication share many of the properties of ordinary arithmetic.

Properties of Vector Addition and Scalar Multiplication

Let \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) be vectors and let \( c \) and \( d \) be scalars. Then the following properties are true.

1. \( \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \)
2. \( (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) \)
3. \( \mathbf{u} + \mathbf{0} = \mathbf{u} \)
4. \( \mathbf{u} + (-\mathbf{u}) = \mathbf{0} \)
5. \( c(d\mathbf{u}) = (cd)\mathbf{u} \)
6. \( (c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u} \)
7. \( c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v} \)
8. \( 1(\mathbf{u}) = \mathbf{u}, \quad 0(\mathbf{u}) = \mathbf{0} \)
9. \( \|c\mathbf{v}\| = |c|\|\mathbf{v}\| \)

Property 9 can be stated as follows: the magnitude of the vector \( c\mathbf{v} \) is the absolute value of \( c \) times the magnitude of \( \mathbf{v} \).
Unit Vectors

In many applications of vectors, it is useful to find a unit vector that has the same direction as a given nonzero vector $\mathbf{v}$.

To do this, you can divide $\mathbf{v}$ by its magnitude to obtain

$$\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||} = \left(\frac{1}{||\mathbf{v}||}\right)\mathbf{v}.$$  

Unit vector in direction of $\mathbf{v}$

Note that $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$.

The vector $\mathbf{u}$ has a magnitude of 1 and the same direction as $\mathbf{v}$.

The vector $\mathbf{u}$ is called a unit vector in the direction of $\mathbf{v}$. 
Example

Find a unit vector in the direction of \( \mathbf{v} = \langle -2, 5 \rangle \) and verify that the result has a magnitude of 1.
The unit vector in the direction of \( \mathbf{v} \) is

\[
\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -2, 5 \rangle}{\sqrt{(-2)^2 + (5)^2}}
\]

\[
= \frac{1}{\sqrt{29}} \langle -2, 5 \rangle
\]

\[
= \left\langle \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right\rangle.
\]
Solution

This vector has a magnitude of 1 because

\[ \sqrt{\left(\frac{-2}{\sqrt{29}}\right)^2 + \left(\frac{5}{\sqrt{29}}\right)^2} = \sqrt{\frac{4}{29} + \frac{25}{29}} \]

\[ = \sqrt{\frac{29}{29}} \]

\[ = 1. \]
Standard Unit Vectors

The unit vectors \( \langle 1, 0 \rangle \) and \( \langle 0, 1 \rangle \) are called the **standard unit vectors** and are denoted by

\[
i = \langle 1, 0 \rangle \quad \text{and} \quad j = \langle 0, 1 \rangle
\]

as shown in the Figure.

(Note that the lowercase letter \( \mathbf{i} \) is written in boldface to distinguish it from the imaginary number \( i = \sqrt{-1} \).)
Unit Vectors

These vectors can be used to represent any vector \( \mathbf{v} = \langle v_1, v_2 \rangle \), as follows.

\[
\mathbf{v} = \langle v_1, v_2 \rangle \\
= v_1 \langle 1, 0 \rangle + v_2 \langle 0, 1 \rangle \\
= v_1 \mathbf{i} + v_2 \mathbf{j}
\]

The scalars \( v_1 \) and \( v_2 \) are called the horizontal and vertical components of \( \mathbf{v} \), respectively.
Unit Vectors

The vector sum

\[ v_1 \mathbf{i} + v_2 \mathbf{j} \]

is called a **linear combination** of the vectors \( \mathbf{i} \) and \( \mathbf{j} \).

Any vector in the plane can be written as a linear combination of the standard unit vectors \( \mathbf{i} \) and \( \mathbf{j} \).
Direction Angles

If \( \mathbf{u} \) is a unit vector such that \( \theta \) is the angle (measured counterclockwise) from the positive \( x \)-axis to \( \mathbf{u} \), the terminal point of \( \mathbf{u} \) lies on the unit circle and you have

\[
\mathbf{u} = \langle x, y \rangle = \langle \cos \theta, \sin \theta \rangle
\]

\[
= (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}
\]

as shown in the Figure.

The angle \( \theta \) is the direction angle of the vector \( \mathbf{u} \).
Suppose that \( \mathbf{u} \) is a unit vector with direction angle \( \theta \).

If \( \mathbf{v} = a\mathbf{i} + b\mathbf{j} \) is any vector that makes an angle \( \theta \) with the positive \( x \)-axis, it has the same direction as \( \mathbf{u} \) and you can write

\[
\mathbf{v} = ||\mathbf{v}||\langle \cos \theta, \sin \theta \rangle
\]

\[
= ||\mathbf{v}|| (\cos \theta)\mathbf{i} + ||\mathbf{v}|| (\sin \theta)\mathbf{j}.
\]
Direction Angles

Because \( \mathbf{v} = a\mathbf{i} + b\mathbf{j} = \|\mathbf{v}\| (\cos \theta)\mathbf{i} + \|\mathbf{v}\| (\sin \theta)\mathbf{j} \), it follows that the direction angle \( \theta \) for \( \mathbf{v} \) is determined from

\[
\tan \theta = \frac{\sin \theta}{\cos \theta}
\]

\[
= \frac{\|\mathbf{v}\| \sin \theta}{\|\mathbf{v}\| \cos \theta}
\]

\[
= \frac{b}{a}. \quad \text{Quotient identity}
\]

\[
= \frac{b}{a}. \quad \text{Multiply numerator and denominator by } \|\mathbf{v}\|. \]
Example

Find the direction angle of each vector.

a. \( \mathbf{u} = 3\mathbf{i} + 3\mathbf{j} \)

b. \( \mathbf{v} = 3\mathbf{i} - 4\mathbf{j} \)
Solution

a. The direction angle is

\[ \tan \theta = \frac{b}{a} = \frac{3}{3} = 1. \]

So, \( \theta = 45^\circ \), as shown in the Figure.

b. The direction angle is

\[ \tan \theta = \frac{b}{a} = \frac{-4}{3}. \]
Solution

Moreover, because \( \mathbf{v} = 3\mathbf{i} - 4\mathbf{j} \) lies in Quadrant IV, \( \theta \) lies in Quadrant IV and its reference angle is

\[
\theta = \left| \arctan\left( -\frac{4}{3} \right) \right|
\]

\[
\approx |-53.13^\circ|
\]

\[
= 53.13^\circ.
\]

So, it follows that
\[
\theta \approx 360^\circ - 53.13^\circ = 306.87^\circ,
\]

as shown in the Figure.
Example

Find the component form of the vector that represents the velocity of an airplane descending at a speed of 150 miles per hour at an angle 20° below the horizontal, as shown in the Figure.
Solution

The velocity vector $\mathbf{v}$ has a magnitude of 150 and a direction angle of $\theta = 200^\circ$.

$$\mathbf{v} = ||\mathbf{v}|| (\cos \theta) \mathbf{i} + ||\mathbf{v}|| (\sin \theta) \mathbf{j}$$

$$= 150 (\cos 200^\circ) \mathbf{i} + 150 (\sin 200^\circ) \mathbf{j}$$

$$\approx 150 (-0.9397) \mathbf{i} + 150 (-0.3420) \mathbf{j}$$

$$\approx -140.96 \mathbf{i} - 51.30 \mathbf{j}$$

$$= \langle -140.96, -51.30 \rangle$$
Solution

You can check that $\mathbf{v}$ has a magnitude of 150, as follows.

$$
\|\mathbf{v}\| \approx \sqrt{(-140.96)^2 + (-51.30)^2}
$$

$$
\approx \sqrt{19,869.72 + 2631.69}
$$

$$
= \sqrt{22,501.41}
$$

$$
\approx 150
$$