2.3. SOLVING TRIGONOMETRIC EQUATIONS
What you should learn

• Use standard algebraic techniques to solve trigonometric equations.

• Solve trigonometric equations of quadratic type.

• Solve trigonometric equations involving multiple angles.

• Use inverse trigonometric functions to solve trigonometric equations.
Solving a trigonometric equation

- *Isolate* the trigonometric function in the equation
- Consider the graph of the related trigonometric function and find the inverse images.

Remark:
Trigonometric functions are periodic functions. Therefore, there may be infinitely many solutions. Be careful with the conditions in the problem!
Example

Solve the equation $2 \sin x = 1$.

**Solution:**

\[
2 \sin x = 1
\]

\[
\sin x = \frac{1}{2}.
\]

(Divide by 2 on both side to isolate)
**Solution 1:** (Using the graph of sine function)

Refer to the graph of sine function and observe that $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$ in the interval $[0, 2\pi)$. 

![Graph of sine function with marked points](image-url)
Final Answer:

\[ x = \frac{\pi}{6} + 2n\pi \quad \text{and} \quad x = \frac{5\pi}{6} + 2n\pi \]

where \( n \) is an integer. (because sine function is a periodic function with period \( 2\pi \).)
Solution 2 (Use the Unit Circle)

Any angles that are coterminal with $\pi/6$ or $5\pi/6$ will also be solutions of the equation.

$$x = \frac{\pi}{6} + 2n\pi \quad \text{and} \quad x = \frac{5\pi}{6} + 2n\pi$$

where $n$ is an integer.
Example

Solve \( \sin x + \sqrt{2} = -\sin x \).

Solution:

Begin by rewriting the equation so that \( \sin x \) is isolated on one side of the equation.

\[
\sin x + \sqrt{2} = -\sin x \quad \text{Write original equation}
\]

\[
\sin x + \sin x + \sqrt{2} = 0 \quad \text{Add } \sin x \text{ to each side.}
\]

\[
\sin x + \sin x = -\sqrt{2} \quad \text{Subtract } \sqrt{2} \text{ from each side.}
\]
Because $\sin x$ has a period of $2\pi$, first find all solutions in the interval $[0, 2\pi)$.

These solutions are $x = 5\pi/4$ and $x = 7\pi/4$. 

\[2 \sin x = -\sqrt{2}\]

Combine like terms.

\[\sin x = -\frac{\sqrt{2}}{2}\]  

Divide each side by 2.
Finally, add multiples of $2\pi$ to each of these solutions to get the general form

\[ x = \frac{5\pi}{4} + 2n\pi \quad \text{and} \quad x = \frac{7\pi}{4} + 2n\pi \]

where $n$ is an integer.
Trigonometric Equations of Quadratic Type:

- $ax^2 + bx + c = 0$ where $x$ is any trigonometric function
- $ax^2 + by + c = 0$ where $x$ and $y$ are any trigonometric functions.

**Example:**

- $2 \sin^2 x – \sin x – 1 = 0$ Quadratic in $\sin x$
- $\sec^2 x – 3 \sec x – 2 = 0$ Quadratic in $\sec x$
- $2 \cos^2 x – \sin x – 1 = 0$ Combined form
Remark:
\[(\sin x)^2 = \sin^2 x\]

How to solve the equation?

Step 1: Convert the given quadratic form to the quadratic form of one trigonometric function.

Step 2: Factor the quadratic equation or if factoring is not possible, then use the quadratic formula.

Step 3: Solve the equation.
Example

Find all solutions of $2 \cos^2 x - 1 = \cos x$.

Solution:

Please watch ProfRobBob.
Find all solutions of $2 \sin^2 x - \sin x - 1 = 0$ in the interval $[0, 2\pi)$.

**Solution: (Please watch ProfRobBob)**

Begin by treating the equation as a quadratic in $\sin x$ and factoring.

$$2 \sin^2 x - \sin x - 1 = 0$$

$$(2 \sin x + 1)(\sin x - 1) = 0$$
Remark: (The same pattern’s quadratic equation)
Original: \( 2a^2 - a - 1 = 0 \)
Factoring: \( (2a + 1)(a - 1) = 0 \)

Setting each factor equal to zero, you obtain the following solutions in the interval \([0, 2\pi)\).

\[ 2 \sin x + 1 = 0 \quad \text{and} \quad \sin x - 1 = 0 \]

\[ \sin x = -\frac{1}{2} \quad \text{and} \quad \sin x = 1 \]

\[ x = \frac{7\pi}{6}, \frac{11\pi}{6} \quad \text{and} \quad x = \frac{\pi}{2} \]
Example

Find all solutions of \( \cot^2 x + \sqrt{3} \cot x = 0 \).

Solution:
Please watch ProfRobBob.
Example

Find all solutions of $4\sin^2 x - \sin x - 1 = 0$.

Solution:
Please watch ProfRobBob.
Example

Find all solutions of
\[ \cos^2 x - \sin^2 x + \sin x = 0 \] in the interval \([0, 2\pi)\).

Solution:
Please watch ProfRobBob.
Functions Involving Multiple Angles of the forms $\sin ku$ and $\cos ku$

To solve equations of these forms, first solve the equation for $ku$, then divide your result by $k$. 
Example

Solve $2 \cos 2t - 1 = 0$ where $t$ is between 0 and $2\pi$.

Solution:
Please watch ProfRobBob.
Example

Solve $2 \cos 3t - 1 = 0$.

Solution:

$$2 \cos 3t - 1 = 0$$

$$2 \cos 3t = 1$$  \quad \text{Add 1 to each side}

$$\cos 3t = \frac{1}{2}$$  \quad \text{Divide each side by 2.}
In the interval $[0, 2\pi)$, you know that $3t = \pi/3$ and $3t = 5\pi/3$ are the only solutions, so, in general, you have

$$3t = \frac{\pi}{3} + 2n\pi \quad \text{and} \quad 3t = \frac{5\pi}{3} + 2n\pi.$$ 

Dividing these results by 3, you obtain the general solution

$$t = \frac{\pi}{9} + \frac{2n\pi}{3} \quad \text{and} \quad t = \frac{5\pi}{9} + \frac{2n\pi}{3}$$

where $n$ is an integer.
Example

Solve \( \sec^2 x - 2 \tan x = 4 \).

Remark:
This is combined form of quadratic equation. It is important to convert the given equation to the quadratic equation of one trigonometric function. To do that, use Pythagorean identity

\[
1 + \tan^2 x = \sec^2 x
\]
Solution

\[ \sec^2 x - 2 \tan x = 4 \]

\[ 1 + \tan^2 x - 2 \tan x - 4 = 0 \]

\[ \tan^2 x - 2 \tan x - 3 = 0 \]

\[ (\tan x - 3)(\tan x + 1) = 0 \]

Setting each factor equal to zero, you obtain two solutions in the interval \((-\pi/2, \pi/2)\). [Recall that the range of the inverse tangent function is \((-\pi/2, \pi/2)\).]
\[ \tan x - 3 = 0 \quad \text{and} \quad \tan x + 1 = 0 \]

\[ \tan x = 3 \quad \text{and} \quad \tan x = -1 \]

\[ x = \arctan 3 \quad \text{and} \quad x = -\frac{\pi}{4} \]

Finally, because \( \tan x \) has a period of \( \pi \), you obtain the general solution by adding multiples of \( \pi \).
\[ x = \arctan 3 + n\pi \quad \text{and} \quad x = -\frac{\pi}{4} + n\pi \]

where \( n \) is an integer.

You can use a calculator to approximate the value of \( \arctan 3 \).